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A STUDY OF JETS FROM EXPLOSIVE SHAPED CHARGES

R. D. Shelton

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A shaped charge jet, approximated as a right circular cylinder with a uniform velocity gradient, is treated from the energetic and stochastic viewpoints to yield, as a function of material properties, the size distribution of jet particles after jet break-up.

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1. INTRODUCTION

The search for models useful in describing and predicting the break-up of shaped-charge jets has proceeded along several lines. The problem of mass flow in jets can be treated in a continuum mechanics fashion by formulating a set of difference equations which can be solved in a reasonable time by a large computer. When spatially periodic perturbations are introduced into the boundaries of the jet, they are found to grow if they are of the right size.[1] Another approach is to make experimental observations and attempt to use similarity methods[2] and empirical curve fitting to extrapolate from the measured to the unknown. A third approach, and the one to be followed in this study, is to use basic mechanics and simple probability theory to treat the jet as a collection of stretching links. All of these approaches are tied to material properties which are reasonable in concept but difficult to measure during the formation and break-up of the jet

It has been observed that high speed jets, created by the detonation of shaped explosive charges with conical copper liners, tend to break up into small particles of various sizes after about 10^{-4} seconds. Continuous jets with all parts traveling along the same straight line are known to have better penetration capabilities than jets which are broken or dispersed, and there is considerable interest in discovering and controlling those physical parameters which influence the pattern of jet break-up. The purpose of this report is to combine a number of observations with some simple theory and to arrive at a model which contains the pertinent physical parameters and predicts the pattern of the break-up of shaped charge jets.

2. SIMPLE DESCRIPTION OF A JET

By examining consecutive x-ray flash photographs of a jet, it is possible to determine its mass distribution in space and time and to describe it rather simply at a given time as shown in Table I, which will be the basis for the following analysis. Jet properties will of course vary with the design of the shaped charge. The values listed in Table I were obtained by a crude averaging of data from firings[3] of precision 3.3 inch (8.4 cm) shaped charges with copper liners, and are thought to be representative. Because copper is the metal

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[1] P.C. Chou and J. Carleone, Stability of Shaped Charge Jets, Journal of Applied Physics, Vol. 48, No. 10, pg. 4187, October 1977

[2] W. E. Baker, P. Westine and F. T. Dodge, Similarity Methods in Engineering Dynamics, Hayden Book Co., Inc., Rochelle Park, NJ, pg. 196, 1973

[3] J. Simon and R. DiPersio, Shaped Charge Warhead Performance, Transactions of 4th Symposium on Warhead Research, US Naval Ordnance Testing Station, China Lake, CA, Sept 1965

most often used in liners for shaped charges, some properties of copper are listed in Table II.

Table I. Nominal Properties of a Copper Jet 10^{-4} Seconds After Shaped Charge Initiation

Velocity of Tip	7.5 km/sec
Length (V>2.5 km/sec)	50 cm
Diameter	3 mm
Break-up Time	100 μ/sec
Number of Particles	50
Mass in Jet	32 gms

Table II. Nominal Properties of Copper

Density	8.93 gm/cm^3
Sound Velocity	3750 m/sec
Thermal Conductivity	3.94 w/cm°C
Heat of Fusion	207 j/gm
Heat of Vaporization	4730 j/gm
Specific Heat	0.38 j/gm°C
Atomic Weight	63.54/mol
Melting Point	1083°C
Boiling Point	2595°C
Strength	2X10^8 newtons/m^2

For purposes of discussion we will assume, as depicted in Figure 1, that the shaped charge produces at the virtual origin a copper jet which expands uniformly in time so that it can be viewed as a stretching right circular cylinder of constant mass and density. The length L of the jet at time t is given by the equation,

$$L = U_s t = 5.0 \times 10^3 t \qquad (2.1)$$

where U_s is a stretching speed of 5.0 km/sec, the difference in speed between the tip and the tail of the jet. If the jet has a diameter D

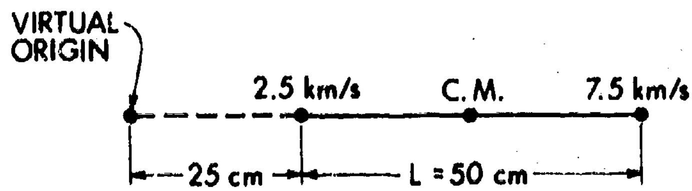

Figure 1. A depiction of the jet resulting from a copper-lined shaped charge. If we assume that the jet started from the virtual origin at time zero, it is 100 microseconds old. The speed of stretching U_s is 5.0 km/sec and the velocity gradient, or strain rate, U_s/L, is 10^4 sec^{-1}.

of 3 mm when it is 50 cm long at 100 μ/sec and if copper has a density of 8.93 gm/cm^3, the mass of the jet is given by the equation,

$$M = \pi D^2 L\rho/4 = 31.56 \text{ gm.} \qquad (2.2)$$

The motion of the jet can be described in terms of the motion of the center of mass and the motion relative to the center of mass by the equation,

$$U(X) = U_c + (X/L)\ U_s; \ - L/2 < X < L/2 \qquad (2.3)$$

where $U(X)$ is the speed of a mass point located by the coordinate X measured from the center of mass, U_c is the speed of the center of mass, L is the length of the jet at time t, and U_s is the difference in speed between the jet tip and tail.

Combining Equations 2.1 and 2.3 yields the result,

$$U(X,t) = U_c + (X/t), \quad -U_s/2 < X/t < U_s/2, \qquad (2.4)$$

where, for our particular jet, both U_c and U_s are 5 km/sec.

3. ENERGY DISTRIBUTION IN THE JET

It is evident that Equation 2.4 can hold only if the stretching of the jet requires no expenditure of energy. The energy of the jet may be divided into two parts, the energy associated with the motion of the center of mass and the energy associated with motion relative to the center of mass, and may be written as

$$E = MU_c^2/2 + (1/2)\int_{X=-L/2}^{X=L/2} \mu u^2(X)\,dX \qquad (3.1)$$

where μ is the mass per unit length in the jet and $u(X)$ is defined from Equation 2.3 by the equation,

$$u(X) = U(X) - U_c = (X/L)U_s. \qquad (3.2)$$

After integration between limits Equation 3.1 becomes

$$E = MU_c^2/2 + MU_s^2/24 \tag{3.3}$$

where we have replaced μ by M/L. Since U_s and U_c are both equal to 5.0 km/sec for our situation, we see that only about 1/13 or 8% of the energy is associated with the stretching motion, and it is this energy which is depleted as work is done in stretching.

Suppose that a uniform jet is divided into n equal segments with the velocity distribution of the Kth segment relative to the laboratory system being given by the equation,

$$U(X) = U_c + (X/L)U_s, \quad A<X<B, \tag{3.4}$$

where

$$A \equiv (-L/2) + (K-1)\ (L/n), \tag{3.5}$$

and

$$B \equiv (-L/2) + K(L/n). \tag{3.6}$$

With these definitions, the location and speed of the center of mass of the Kth particle are given by the equations,

$$X_K = -L/2 + (K-\tfrac{1}{2})\ (L/n) = (A+B)/2, \tag{3.7}$$

and

$$U_K = U_c + [-\tfrac{1}{2} + (K-\tfrac{1}{2})/n]\ U_s = U(X_K). \tag{3.8}$$

The velocity distribution in each segment relative to its center of mass can be written as

$$U'(X'/L) = (X'/L)\ U_s, \quad -L/2n<X'<L/2n \tag{3.9}$$

so that the relative kinetic energy can be written as

$$E'_K = \int_{X=-L/2n}^{X=L/2n} (M/2L)\ (X'/L)^2\ U_s^{\ 2}\ dX' \tag{3.10}$$

$$= (M/2L)\ (U_s/L)^2[X'^3/3]\Bigg|_{-L/2n}^{L/2n} \tag{3.11}$$

$$= MU_s^{\ 2}/24\ n^3 \tag{3.12}$$

From Equation 3.8, the kinetic energy of the Kth particle, less its internal energy relative to its center of mass, can be written as

$$E_K = (M/2n)\ U_K^2 \tag{3.13}$$

$$= (\mu/2n)\ \{U_c + [-\tfrac{1}{2} + (K-\tfrac{1}{2})/n]U_s\}^2 \tag{3.14}$$

Using Equations 3.12 and 3.13 the total kinetic energy of the jet can be written as

$$E = \sum_{K=1}^{K=n} E_K + \sum_{K=1}^{K=n} E_K' \tag{3.15}$$

$$= \sum_{K=1}^{K=n} (M/2n)\ \{U_c + [-\tfrac{1}{2} + (K-\tfrac{1}{2})/n]U_s\}^2 + \sum_{K=1}^{K=n} MU_s^2/24n^3 \tag{3.16}$$

Using the identities,

$$\sum_{K=1}^{K=n} a = na,\ a \equiv \text{any constant}, \tag{3.17}$$

$$\sum_{K=1}^{K=n} K = (\tfrac{1}{2})n(n+1), \tag{3.18}$$

$$\sum_{K=1}^{K=n} K^2 = (1/6)(n)(n+1)(2n+1), \tag{3.19}$$

Equation 3.16 can be simplified to the form,

$$E = MU_c^2/2 + (M/24)U_s^2[1-(1/n^2)] + (M/24)U_s^2/n^2 \tag{3.20}$$

$$= MU_c^2/2 + (M/24)U_s^2\,. \tag{3.21}$$

Since n segments contribute to the last term of Equation 3.21, the contribution of each segment to the term is

$$E_n = (1/n)(M/24)(U_s^2/n^2) = (M/24)U_s^2/n^3, \tag{3.22}$$

in agreement with Equation 3.12. Equation 3.20 states that the n segments contribute to the total energy by three terms. The first term is associated with the motion of the mass M at speed U_c. The second term arises from the motions of the centers of mass of the several segments relative to the center of mass of the jet. The last term arises from the internal motion of each segment relative to its center of mass. The total energy of the jet is of course independent of the manner of its division, as illustrated by Equations 3.20 and 3.21. These equations can be written in terms of energy per unit mass as

$$E/M = U_c^2/2 + (1/24)U_s^2[1-(1/n^2)] + (1/24)U_s^2/n^2 \tag{3.23}$$

and

$$E/M = U_c^2/2 + (1/24)U_s^2. \tag{3.24}$$

The above calculations illustrate that, with a given stretching speed, the energy available to stretch an isolated segment of jet, after a partial break-up has occurred, varies strongly with length. It should be stated further that, if a jet in the process of stretching suddenly breaks into n pieces, the total kinetic energy is still the same, but the energy available per unit mass to stretch the jet (or its several pieces) has fallen from $U_s^2/24$ to $U_s^2/24n^2$, as indicated by Equations 3.23 and 3.24.

4. RELATIVE ENERGY CONSUMED IN STRETCHING

The element of energy dE consumed in stretching a rod an elemental distance dX is given by the equation,

$$dE = F\,dX, \tag{4.1}$$

where F is the applied force. By the definition of stress S as force per unit area and strain de as elongation per unit length, Equation 4.1 can be written as

$$dE = (F/A)\ A\ X\ (dX/X) = V\ S\ de, \qquad (4.2)$$

where $S \equiv F/A$, $de \equiv dX/X$ and $V = AX$. (4.3)

Then, $E/V = \int S de$.

For purposes of illustration, suppose that the jet from a shaped charge is assumed to start from a right circular cylinder of some initial length L_o and that it continues to stretch uniformly until it breaks up at some length L_f. From the definition of de Equation 4.3 can be written as

$$E/V = \int_{L_o}^{L_f} S\ (dX/X) = S \ln (L_f/L_o) \qquad (4.4)$$

where S has been assumed to be a constant parameter during the stretching process.

Suppose that a jet exists initially as a right circular cylinder with a length to diameter ratio (L_o/D_o) of unity. The volume V of the jet can be written as

$$V = M/\rho = \pi\ D_o^2\ L_o/4 = \pi\ L_o^3/4, \qquad (4.5)$$

so that, using values for the mass M and density ρ from Tables I and II, the initial length L_o is calculated as

$$L_o = (4M/\pi\rho)^{1/3} = 1.67 \text{ cm}. \qquad (4.6)$$

Using Equations 3.24 and 4.4 and assuming that all the internal kinetic energy of the jet is used in stretching and that there is no jet break-up,

$$U_s^2/24 = (S/\rho) \ln (L_f/L_o) \qquad (4.7)$$

and

$$L_f = L_o \text{EXP}[\rho U_s^2/24S]. \qquad (4.8)$$

$$= 1.64 \, \text{EXP}[8.93 \times 10^3 \cdot 5000^2/(24 \cdot 2\times 10^8)] \quad (4.9)$$

$$= 2.64 \times 10^{20} \text{ cm.} \quad (4.10)$$

This result, though perhaps unrealistic, illustrates that the internal mechanical energy of a jet is sufficient to stretch it until it breaks. If we had assumed a break up length of 50 cm, the work done, from Equation 4.4, would have been

$$E/\rho V = (2 \times 10^8/8.93 \times 10^3) \ln (50/1.67) \quad (4.11)$$

$$= 76.1 \text{ j/gm.}$$

This is only about 7% of the available internal kinetic energy, given from the last term of Equation 3.24 as

$$E/M = U_s^2/24 = 1042 \text{ j/gm.} \quad (4.12)$$

From Table II the specific heat of copper is 0.38 j/gm°C and the heat of fusion is 207 j/gm, so that the work calculated in Equation 4.11 is rather modest compared to the energy required to heat copper from 500°C, for example, and melt it. Since the stress S, assumed for the foregoing calculations to be that of cold copper, is expected to decrease markedly as copper approaches the melting point, there is little likelihood that work done on a stretching jet will cause it to melt if it was a solid at some time during the stretching process.

It will now be assumed, in agreement with observations, that the stretching jet will break rather than stretch indefinitely. In physical terms it can be said that there is apparently some threshold of work by stretching which the jet can endure before it breaks up into segments. Each segment has a velocity gradient U_s/L and will continue to stretch until it breaks further or relief waves propagate from the free ends and the velocity gradient is removed. The energy associated with the velocity gradient persists until it is expended by work done in stretching. The energy available to stretch a segment of length ℓ is given from Equation 3.22 as

$$E = (M/24)U_s^2/n^3 = (1/24) \, (M\ell/L) \, (U_s/L)^2 \ell^2 \quad (4.13)$$

where n has been replaced by L/ℓ and factors have been associated to define the mass $M\ell/L$ of the segment and the velocity gradient U_s/L.

The internal energy per unit mass in the segment is then

$$E/M(\ell/L) = (1/24)(U_s/L)^2\ell^2 . \qquad (4.14)$$

Suppose, for example, that the first breaks occur at some energy threshold $(E/M)_o$ and that subsequent breaks occur if the segment has enough relative energy to push the work done on the segment beyond some upper limit $(E/M)_m$.

Equating the additional work needed for further break-up to the energy available, as given by Equation 4.14, we obtain

$$(1/24)(U_s/L)^2\ell^2 = (E/M)_m - (E/M)_o \equiv \Delta E. \qquad (4.15)$$

If we argue that the segment will not break further if

$$(1/24)\ (U_s/L)^2\ell^2 < \Delta E, \qquad (4.16)$$

then the equation,

$$\ell < [24\ (\Delta E)/(U_s/L)^2]^{1/2} \qquad (4.17)$$

defines the longest segment that can exist without further break-up. By means of Equation 4.4 and 4.15

$$\Delta E = (S/\rho)\ \ln\ (L_m/L_o), \qquad (4.18)$$

where L_o is the length of the segment when break-up starts and L_m is its length when it has used all of its internal kinetic energy in stretching, Equation 4.17 can be written with Equation 4.18 to yield

$$\ell < [24\ (S/\rho)\ \ln\ (L_m/L_o)]^{1/2}/(U_s/L) \qquad (4.19)$$

Assuming a 5% stretch ($L_m/L_o = 1.05$) and nominal values of strength S, density ρ, and velocity gradient U_s/L from Tables I and II and Figure 1, we find that

$$\ell < 1.62\ \text{cm} \qquad (4.20)$$

Data, obtained by Jameson and exhibited in Figure 7, show an estimated maximum length of 1.98 cm.

One way of looking at the energy or time interval associated with jet break-up is to balance the energy available for stretching a jet particle against the work done in stretching. For example, a particle of length ℓ, velocity gradient U_s/L, and mass M will have available for stretching at most the energy

$$E = (M/L)(U_s/L)^2(\ell^3/24) \tag{4.21}$$

as given by Equation 4.13. The energy per unit volume is therefore

$$E\rho/[M(\ell/L)] = \rho\ell^2(U_s/L)^2/24 \tag{4.22}$$

If S is the strength of the material and the elongation Δe in time Δt is given by the equation

$$\Delta e = \Delta\ell/\ell = U_s(\ell/L)\ \Delta t/\ell, \tag{4.23}$$

the work done per unit volume by stretching in time Δt is given from Equation 4.2 as

$$dE/V = S\Delta e = SU_s(\ell/L)\ \Delta t/\ell. \tag{4.24}$$

If we equate the energy available to the work done in time Δt (Equations 4.22 and 4.24)

$$\rho U_s^2\ (\ell/L)^2/24 = SU_s\ \Delta t/L \tag{4.25}$$

and

$$\Delta t = \rho U_s L(\ell/L)^2/24\ S = \rho(U_s/L)\ell^2/24\ S. \tag{4.26}$$

If we use the value of ℓ from Equation 4.20 we find that (4.27)

$$\Delta t = 4.88 \times 10^{-6}\ \text{sec.}$$

From purely kinematic considerations using Equation 2.1

$$\Delta t = t\ (\Delta L/L) = 5 \times 10^{-6}\ \text{sec.} \tag{4.28}$$

The time internal for Δt required for a relief wave to travel across a 1.62 cm segment is given by

$$\Delta t = \ell/a = 1.67 \times 10^{-2}/3750 = 4.45 \times 10^{-6} \text{ sec.} \qquad (4.29)$$

As will be shown later, this transit time for a relief wave is intimately related to particle size distribution.

The point of the preceding discussion is that maximum particle length (Equation 4.20) and relief wave speed (Equation 4.29) can be consistently related by a simple stretching model, which includes work done in stretching.

An interesting consequence of Equation 4.17 is that, if the amount of work permissible by stretching becomes small in the time interval between when break-up starts and when it ends, the jet will break into very small pieces. The theory to support this initial conclusion will be developed later.

It can be argued that the assumptions leading to Equation 4.20 and its fair agreement with experiment are ad hoc and fortuitous. It will be shown later that the same assumptions lead to a reasonable distribution in size for the fragments. Because of the square root and logarithmic dependence of ℓ on L_m/L_o, the result is not particularly sensitive to fairly large variations of L_m/L_o. For example if L_m/L_o had been 10, the result would have been 11.0 cm. This variation in length from 1.62 to 11 cm represents a factor of 200 in the amounts of stretch assumed.

The velocity gradient (or strain rate) U_s/L is probably the most easily measured of the quantities in Equation 4.19. The density ρ is not expected to vary greatly. The strength S, on the other hand, is somewhat of a mystery because of the apparent tendency of copper to keep its strength better as the strain rate is increased, as shown in Figure 2.

The preceding discussion can be summarized as follows. The jet will either stretch forever (Equation 4.10) or it will break up. If it breaks up and its time of break-up is determined by the amount of work done in stretching it, there will be some threshold (of stretch, time, or work) at which breakage begins.

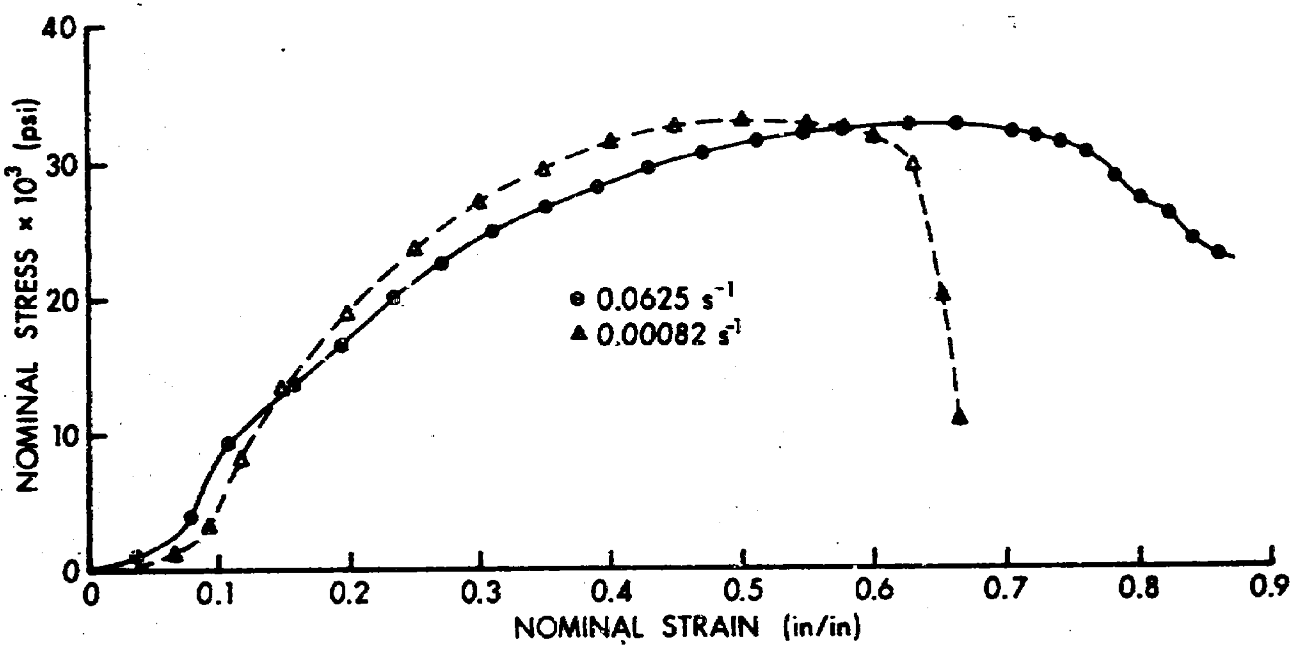

Figure 2. Nominal stress-strain curve for electrolytic tough pitch copper, supplied by Stanley K. Golaski, BRL 1977.

If breakage occurs, each segment has a velocity distribution and therefore a kinetic energy relative to its center of mass, and this kinetic energy must be consumed in further stretching of the segment. If there are further potentially weak points in the segment, like the ones which produced the segment in the first place, the segment will subdivide further as more stretching is done. This process will discontinue rather abruptly because of the sensitivity of the energy available for stretching to the segment length (Equation 4.14). This process can be modeled simply (Equation 4.19) to yield an estimate of the upper limit for the length of the segments following break-up.

5. ELASTIC AND INELASTIC BEHAVIOR OF A LONG RIGHT CIRCULAR CYLINDER WITH A LONGITUDINAL VELOCITY DISTRIBUTION

In order to better understand the behavior of a jet particle with a velocity gradient and two free ends, it is useful to examine the elastic response of a long right circular cylinder to a velocity gradient as an initial condition. We can write the longitudinal displacement $y(X,t)$ from equilibrium at time t of a point located at a distance X from the center of mass in terms of the Fourier series,

$$y(X,t) = \sum_{n=0}^{n=\infty} b_n \sin[(n\pi at/L) + \epsilon_n] \sin[(n\pi X/L) + \delta_n] \qquad (5.1)$$

where L is the length of the cylinder, a is the speed of sound in the cylinder and b_n, ϵ_n, and δ_n are constants determined by initial and boundary conditions. We will assume that, at zero time, we have

$$y(X,0) = 0, \quad -L/2 \leq X \leq L/2, \qquad (5.2)$$

and

$$\dot{y}(X,0) = U_s X/L, = RX, \quad -L/2 \leq X \leq L/2, \qquad (5.3)$$

i.e. that the rod is initially unstretched but has a uniform velocity gradient. The dot over the y indicates partial differentiation with respect to time, U_s is the stretching velocity defined earlier, and

$$R \doteq U_s/L \qquad (5.4)$$

is the strain rate.

The compatibility of Equations 5.1 and 5.2 requires that

$$\epsilon_n = 0. \tag{5.5}$$

Differentiating Equation 5.1 with respect to time t and using Equation 5.5 gives the result,

$$\dot{y}(X,t) = \sum_{n=0}^{n=\infty} (n\pi a/L) b_n \cos(n\pi a t/L) \sin[(n\pi X/L)+\delta_n], \tag{5.6}$$

so that, from Equations 5.3 and 5.6,

$$\dot{y}(X,0) = \sum_{n=0}^{n=\infty} (n\pi a/L) b_n \sin[(n\pi X/L)+\delta_n] = RX \tag{5.7}$$

The Fourier expansion for X in the range $-L/2 < X < L/2$ can be written[4] in the form,

$$X = (4L/\pi^2) \sum_{n=1}^{n=\infty} [(-1)^{n-1}/(2n-1)^2] \sin[(2n-1)\pi X/L] \tag{5.8}$$

Comparison of Equations 5.7 and 5.8 shows that it is possible to write the coefficients of the series in the form

$$\delta_n = 0 \tag{5.9}$$

$$b_n = (4L^2R/\pi^3 a)\ (-1)^{n-1}/(2n-1)^3 \tag{5.10}$$

Using Equations 5.5, 5.9, and 5.10, Equation 1 can be written as

$$y(X,t) = [4L^2R/\pi^3 a] \sum_{n=1}^{n=\infty} [(-1)^{n-1}/(2n-1)^3] \sin[(2n-1)\pi a t/L] \sin [(2n-1)\pi X/L] \tag{5.11}$$

[4] Handbook of Tables for Mathematics 4th Edition. The Chemical Rubber Cor. 18901 Cranwood Parkway, Cleveland, Ohio 44123, pg. 611 formula 7, 1964.

The velocity and acceleration are computed from Equation 5.11 as

$$\dot{y}(X,t) = (4LR/\pi^2) \sum_{n=1}^{n=\infty} [(-1)^{n+1}/(2n-1)^2]\cos[(2n-1)\pi at/L]\sin$$

$$[(2n-1)\pi X/L] \qquad (5.12)$$

and

$$\ddot{y}(X,t) = (4Ra/\pi) \sum_{n=1}^{n=\infty} [(-1)^{n}/(2n-1)]\sin[(2n-1)\pi at/L]\sin[(2n-1)\pi X/L] \qquad (5.13)$$

Since each element of the cylinder is accelerated according to Newton's second law, we can write for the mass between X and X+dX the equation, $dT = \mu\ddot{y}dX$, where T is the tension in the cylinder and μ is the mass per unit length, so that

$$\int_{L/2}^{X} dT = \mu \int_{L/2}^{X} \ddot{y}\, dX. \qquad (5.14)$$

Using Equation 5.13 and the fact that the tension at the end of the cylinder is zero we have

$$T(X,t) = [4Ra\mu/\pi] \int_{L/2}^{X} dX \sum_{n=1}^{n=\infty} [(-1)^{n}/\sin[(2n-1)\pi at/L]\sin[(2n-1)\pi X/L] \qquad (5.15)$$

$$= [4Ra\mu L/\pi^2] \sum_{n=1}^{n=\infty} [(-1)^{n+1}/(2n-1)^2]\sin[(2n-1)\pi atL]\cos[(2n-1)$$

$$\pi X/L] \Big|_{L/2}^{X}. \qquad (5.16)$$

Using the fact that the cosine terms vanish when X=L/2 and the definitions

$$S \equiv T/\pi r^2 \equiv \text{stress} \qquad (5.17)$$

$$\rho \equiv M/(\pi r^2 L) \qquad (5.18)$$

$$\mu \equiv M/L \tag{5.19}$$

where M is the mass of the cylinder and r is its radius, we have finally

$$S(X,t) = [4Ra\rho L/\pi^2] \sum_{n=1}^{n=\infty} [(-1)^{n+1}/(2n-1)^2]\sin[(2n-1)\pi at/L]\cos[(2n-1)\pi X/L] \tag{5.20}$$

On the basis of Equation 5.20 a number of conclusions may be drawn. At small times, i.e. when

$$\sin[(2n-1)\pi at/L] \approx [(2n-1)\pi at/L], \tag{5.21}$$

Equation 5.20 reduces to

$$S(X,t) = [4\rho Ra/\pi^2] \sum_{n=1}^{n=\infty} [(-1)^{n+1}/(2n-1)](\pi a/L)t\cos[(2n-1)\pi X/L]. \tag{5.22}$$

If we use the series identity,

$$\pi/4 = \sum[(-1)^{n+1}/(2n-1)]\cos[(2n-1)\pi X/L], \tag{5.23}$$

Equation 5.22 reduces to

$$S(X,t) = (\rho Ra^2 t) \tag{5.24}$$

Equation 5.24 is an expression of our expectation that a uniform velocity gradient produces a uniform stretching until a relief wave can propagate from the ends of the cylinder. The time required for the uniform stretching to exceed the elastic limit of copper in our "standard jet" is estimated from Equation 5.24 as

$$\begin{aligned} t &= S/(\rho Ra^2) \\ &= 0.16\text{x}10^{-6}\text{sec.} \end{aligned} \tag{5.25}$$

We see that the strain in a stretching jet is sufficient to insure plastic flow from jet formation to break-up. Plastic flow reduces the velocity a at which signals propagate so that what happens at one point in the jet does not communicate rapidly to neighboring points. Therefore the jet continues to stretch in a manner determined by the local velocity gradient or strain rate.

If the deformation of the cylinder were totally elastic, we would expect maximum strain when every term in Equation 5.20 is positive or

$$\pi at/L = \pi/2, \tag{5.26}$$

or

$$t = L/2a. \tag{5.27}$$

This is the time for a sound wave to travel from the end to the center of an elastic cylinder. In this case Equation 5.20 becomes

$$S(X,L/2a) = [4\rho RaL/\pi^2] \sum_{n=1}^{n=\infty} [1/(2n-1)^2]\cos[(2n-1)\pi X/L] \tag{5.28}$$

We now use the fact that the Fourier series for $[(L/2)-X]$ in the range $-L/2 \leq X \leq L/2$ is given by

$$L/2-X = (4L/\pi^2) \sum_{n=1}^{n=\infty} [1/(2n-1)^2]\cos[(2n-1)\pi X/L], \tag{5.29}$$

so that Equation 5.28 becomes

$$S(X,L/2a) = \rho Ra[L/2 - X] \tag{5.30}$$

This agrees with our boundary condition that the stress vanishes at the ends and our expectation that the stress is a maximum at the center of the cylinder.

The preceding discussion of the jet as an elastic right circular cylinder illustrates a number of points which are expected but which are perhaps more believable if demonstrated by a standard mechanical analysis. The tension along the jet is uniform (Equation 5.25) until the elastic limit is exceeded, and is expected to remain so until the jet begins to break-up, if perturbations are expected to propagate poorly in the plastically stretching jet. With a given velocity gradient, the maximum stress available (Equation 5.30) is proportional to the product of length, velocity gradient, wave velocity, and density. This agrees with our earlier calculations, based on kinetic energy considerations, that longer segments will stretch more because they have more energy per unit mass (Equation 4.14). Since the tension is greatest at the middle of the jet (Equation 5.30) a relieved segment (one that has lost its velocity gradient) will be stretched more in the middle and have a tendency to be bigger at the ends, where stretching is least.

If we assign a maximum stress level s, Equation 5.30 permits a calculation of the distance d from the end at which this stress in exceeded and plastic deformation begins:

$$d \equiv (L/2-X) = S/\rho Ra$$
$$= 2x10^8/(8.93\cdot 10^3\cdot 10^4\cdot 3750) = 0.6mm \quad (5.31)$$

This means that segments above about 1mm will use almost all of their kinetic energy, relative to the center of mass, in causing plastic deformation.

As a matter of interest, whether or not it has physical reality in the case of the actual jet, the relief wave starting from the free end of the jet should cause the jet to stretch less in the region near the end and hence insure that the tip particle will have the largest diameter and will have a greater length than average.

If we imagine a jet to start as a right circular cylinder and to break-up in 10^{-4} seconds with open spaces of one cm roughly equal to the length of the particles, we can estimate crudely the magnitude of the forces involved. The jet has roughly 0.6 gms/cm mass, and a mass of 0.3 gm must move a distance of 0.25 cm on the average to vacate a space of 0.5 cm. Using Newtons second Law, we derive the formula,

$$F = 2MS/t^2 \quad (5.32)$$

where F is the average applied force which must be exerted to move a mass m a distance X in time t. Using the above numbers,

$$F = 2\cdot 0.3x10^{-3}x0.5x10^{-2}/(10^{-4})^2 = 300n \quad (5.33)$$

The surface tension ν of molten copper is 1.103 n/m, so that its contribution to the tension T in a jet of 3 mm diameter D is computed to be

$$T = \pi D\nu = 0.01n. \quad (5.34)$$

This means that the surface tension is too small by a factor of $3x10^4$ to produce the observed motion, and that the jet is unlikely to have broken up because it was a liquid cylinder unstable under forces of surface tension. On the other hand, the strength of copper is roughly $2x10^8$ n/m^2, so that the tension in the jet if it is a cold solid is $1.4x10^3$n. A 20% variation in this tension would be required to produce the specified displacements in 10^{-4} secs, so that even a sizeable instability might not be sufficient to produce the necessary action unless it grew rapidly from the beginning.

6. A SIMPLE STOCHASTIC MODEL FOR JET BREAK-UP

With the preceding theoretical framework it is now possible to discuss jet break-up in a more quantitative fashion. It will be assumed that the copper jet stretches as a solid until enough work is done on it to cause it to break into pieces. The distribution in length of the pieces will be determined by the energy available to stretch a piece further after it is formed, the distribution of weak points or inhomogenities in the jet, and the properties of the material in general.

The first part of the determination is given by Equation 4.14,

$$E/[M(\ell/L)] = (1/24)(U_s/L)^2\ell^2, \qquad (6.1)$$

which relates the specific kinetic energy relative to the center of mass of a segment to the velocity gradient (U_s/L) and length ℓ. We are justified in keeping Equation 6.1 while the jet stretches and begins to break-up because we have shown following Equation 4.11 that only about 7% of the jet energy is consumed in stretching and have argued that the jet stretches uniformly with little communication up and down the jet because of the rapid rate of plastic deformation.

The second part of the determination, namely the distribution of weak points in a jet, is something easy to believe but difficult to demonstrate, other than by noting that the jet does break-up into a somewhat irregular pattern.

Referring to Figure 3, suppose that the probability, that any segment of length ΔX will break during the time interval between t and $t+\Delta t$, is given by the expression,

$$P(t)\ \Delta X\ \Delta E = B(t)\ \Delta X\ \Delta E, \qquad (6.2)$$

where ΔE is the work done on the segment by stretching. The assumptions contained in this equation are (1) that the probability of a weak point being in the segment is proportional to its length and (2) that the probability of its breaking is proportional to the work done on it by stretching.

Since work proceeds as a function of time as the jet stretches, Equation 6.2 can be written in the more convenient form,

$$p(t)\ \Delta X\ \Delta t = b(t)\ \Delta X\ \Delta t, \qquad (6.3)$$

where

$$t_o < t < t_m. \qquad (6.4)$$

Here t_o is the time when break-up starts and t_m is the time when it ends.

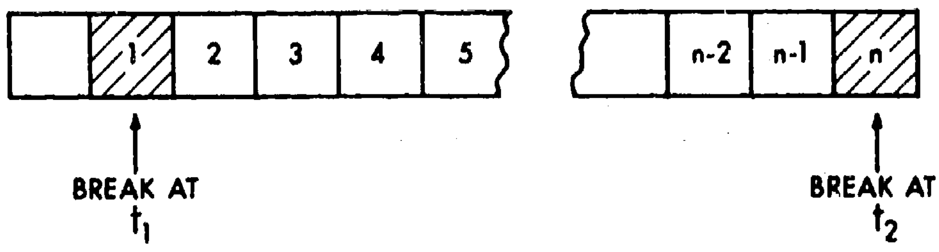

Figure 3. Calculation of the probability that n segments of total length L will hold together. Breaks are postulated for segments i and n, and the probability that no other breaks occur before a relief wave arrives is calculated.

The inclusion of the limits on the time t reflects the experimental observation that, in stretching a number of like samples of copper, the elongations at which breaks occur are bunched between upper and lower limits which are rather loosely spaced. Between these limits, the work done is roughly proportional to the elapsed time Δt. The parameter $b(t)$ is characterized by the material and its condition at time t.

Suppose that the time interval between t_o and t_i is divided into M equal intervals of length Δt and that the jet is also divided into imaginary segments. The probability P_{ik} that the ith segment of length $(\Delta X)_i$ will survive the Kth time interval $(\Delta t)_K$ without a break is given from Equation 6.3 as

$$p_{ik} = 1 - b(t)\ (\Delta X)_i\ (\Delta t)_K, \tag{6.5}$$

i.e. the survival probability p_{ik} is unity minus the probability of a break, given by Equation 6.3.

The total survival probability of the segment is the product of all the individual survival probabilities, i.e.,

$$p_i(t) = \prod_{K=1}^{K=M} [1-b(t)(\Delta X)_i(\Delta t)_K] \tag{6.6}$$

Taking the logarithm of both sides yields the result,

$$\ell n\ p_i(t) = \sum_{K=1}^{K=M} \ell n[1-b(t)(\Delta t)_i(\Delta t)_K] \tag{6.7}$$

Since

$$\lim_{y\to 0} \ell n\ (1-y) = -y \tag{6.8}$$

and the products $b(t)(\Delta X)_i(\Delta t)_K$ will become arbitrarily small as $(\Delta X)_i$ and $(\Delta t)_K$ approach zero, it is a good approximation to write

$$\ell n\ p_i(t) = -\sum_{K=1}^{K=M} b(t)(\Delta X)_i(\Delta t)_K. \tag{6.9}$$

The identification,

$$\sum_{K=1}^{K=M} b(t)(\Delta t)_K = \int_{t_o}^{t_i} b(t)dt, \tag{6.10}$$

permits Equation 6.9 to be written as

$$\ell n\ p_i(t) = -(\Delta X)_i \int_{t_o}^{t_i} b(t)dt \tag{6.11}$$

Equation 6.11 is a statement of the probability that the ith segment of length $(\Delta X)_i$ will survive if a relief wave arrives at time t_i. It is assumed that the process of jet break-up is as shown in Figure 4 and that a segment is subject to breakage until it is relieved by a relief wave propagating from a nearby break.

In a like manner it can be stated that, if a rod of length L is to survive until time t_m, each sub-element $(\Delta X)_i$ must be relieved before that time. Suppose that the rod is divided into n segments of equal length, as shown in Figure 3, and that the rod is formed by a break on the left at time t_1 and a break on the right at time t_2. Suppose that a relief wave travels from the left break with a speed U(X) and arrives at the ith segment at time t_i. The survival probability of the ith segment is given from Equation 6.11 as

$$p_i(t_i) = \mathrm{EXP}[-(\Delta X)_i \int_{t_o}^{t_i} b(t)dt \tag{6.12}$$

where

$$t_i = t_1 + \int_0^{X_i} dX/U(X). \tag{6.13}$$

The distance X_i is measured from the left end of the rod to the ith segment, and U(X) is the speed at which the relief wave travels from the left break. The integral in Equation 6.13 is just the time required for the relief wave to travel from the left break to the ith segment. In general, the relief waves from the right and left will meet at some time t_u and at some point X_{12} such that

$$t_u = t_1 + \int_0^{X_{12}} dX/U(X) = t_2 + \int_0^{X_{12}'} dX'/U(X'), \tag{6.14}$$

where

$$X_{12}' \equiv L - X_{12} \tag{6.15}$$

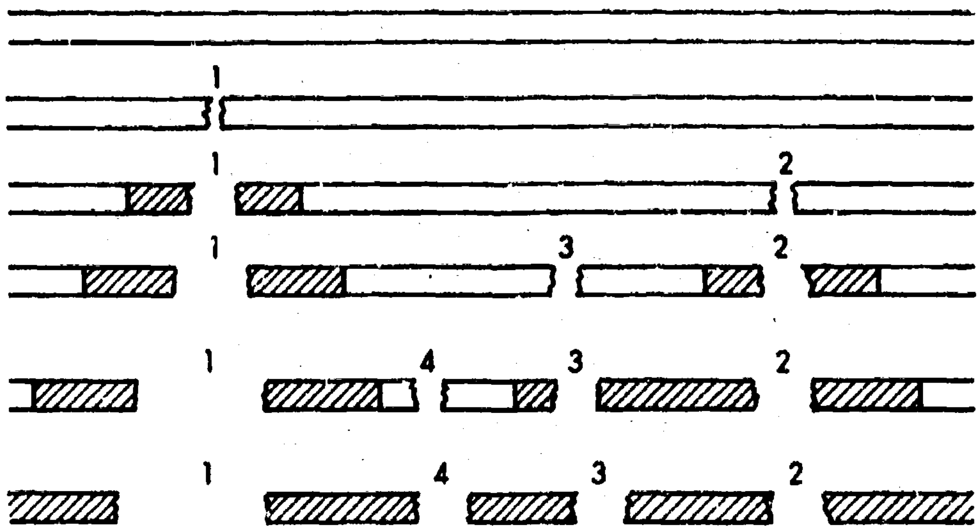

Figure 4. Break up of a jet. Break points are numbered. Relieved areas are shown by cross hatch. Each segment continues to stretch until it is either broken or relieved by a relief wave traveling from a free end.

From the above considerations and reference to Figure 3, the probability that a break will occur on the left between the times t_1 and $t_1 + (\Delta t)_1$ and on the right between the times t_2 and $t_2 + (\Delta t)_2$ and that the intervening segments will survive until reached by a relief wave, after which they are sure to survive, can be written as

$$P(L,t_1,t_2)(\Delta t)_1(\Delta t)_2 = \prod_{j=1}^{j=M} p_j \prod_{K=1}^{K=n} p_K \, b(t_1)(b(t_2)(\Delta t)_1(\Delta t)_2 \qquad (6.16)$$

In the above expression, $b(t_1)(\Delta t)_1$ is the probability per unit length that the first segment will break between the time t_1 and $t_1 + (\Delta t)_1$ and $b(t_2)(\Delta t)_2$ is the probability per unit length that the last segment will break between the times t_2 and $t_2 + (\Delta t)_2$. For Equation 6.16 to be a true probability equation, it should be multiplied on both sides by $(\Delta X)_1(\Delta X)_2$ to include the lengths of the segments where the breaks occur. However, because only a relative probability is sought and because the location of the breaks is immaterial to the extent that any other two breaks separated by the same distance would yield the same results, the factor $(\Delta X)_1(\Delta X)_2$ is ignored to save writing and space. The first product,

$$\prod_{j=1}^{j=M} p_j, \qquad (6.17)$$

is the probability that the left part of the rod of length L, defined by

$$0 < X < X_{12}, \qquad (6.18)$$

will survive, and the second product is the probability that the right part will survive. Total survival probability is the product of the survival probabilities of the individual parts.

For convenience of analysis, the left hand side defined by Equation 6.18 is divided into M equal segments of length ΔX and the right hand side is divided into n equal segments y length ΔX so that

$$(M+n)\Delta X = L \qquad (6.19)$$

From Equations 6.12 and 6.13,

$$p_j = \mathrm{EXP}[-\Delta X \int_{t_o}^{t_j} b(t)dt]; \qquad (6.20)$$

$$p_K = \mathrm{EXP}[-\Delta X \int_{t_o}^{t_K} b(t)dt]; \qquad (6.21)$$

$$t_j = t_1 + \int_0^{X_j} dy/U(y) = t_1 + f(X_j); \qquad (6.22)$$

$$t_K = t_2 + \int_0^{L-X_K} dy'/U(y') = t_2 + f(L-X_K). \qquad (6.23)$$

By way of explanation, Equation 6.20 states the survival probability of the jth segment relieved from the left by a relief wave starting some time between t_1 and $t_1 + (\Delta t)_1$ and arriving at time t_j. Equation 6.21 computes t_j in terms of the initiation of the relief wave at time t_1 and its travel time to the jth segment located at X_j. Combining Equations 6.16-6.23 and taking the logarithm gives the result,

$$\ell n\, P(L,t_1,t_2) = \ell n[b(t_1)b(t_2)] - \Delta X \sum_{j=1}^{j=M} \int_0^{t_j} b(t)dt - \Delta X \sum_{K=1}^{K=n} \int_0^{t_K} b(t)dt. \quad (6.24)$$

In the limit as ΔX becomes arbitrarily small

$$\Delta X \sum_{j=1}^{j=M} \int_{t_o}^{t_j} b(t)dt = \int_{t_o}^{X_{12}} dX \int_{t_o}^{t(t_1,X)} b(t)dt, \text{ etc}, \qquad (6.25)$$

so that Equation 6.24 can finally be written as

$$\ell n\, P(L,t_1,t_2) = \ell n[b(t_1)b(t_2)] - \int_0^{X_{12}} dX \int_{t_o}^{t(t_1,X)} b(t)dt - \int_0^{X_{12}'} dX \int_{t_0}^{t(t_2,X)} b(t)dt \qquad (6.26)$$

The integration for the left side extends from time t_o, when breaks can begin, to the time $t(t_1,X)$ when the relief wave arrives, where from Equation 6.22

$$t(t_1,X) = t_1 + \int_0^X dy/U(y). \tag{6.27}$$

Likewise,

$$t(t_2,X) = t_2 + \int_0^{L-X} dy'/U(y'). \tag{6.28}$$

The integration over X, for example, continues from the left end to the point X_{12} where the relief waves meet. Equation 6.14 determines the meeting point. Finally

$$P(L) = \int\int dt_1\, dt_2\, P(L,t_1 t_2). \tag{6.29}$$

To exercise this theoretical structure, it is worthwhile to make the simplest possible assumptions, namely that

$$b(t) = b, \tag{6.30}$$

where b is a constant and that relief waves travel with a constant speed a, i.e.

$$U(X) = a. \tag{6.31}$$

Equation 6.14 then becomes

$$t_u = t_1 + \int_0^{X_{12}} dX/a = t_2 + \int_0^{L-X_{12}} dX/a \tag{6.32}$$

$$= t_1 + X_{12}/a = t_2 + (L-X_{12})/a \tag{6.33}$$

so that

$$X_{12} = [a(t_2 - t_1) + L]/2. \tag{6.34}$$

From Equations 6.27 and 6.28,

$$t(t_1,X) = t_1 + X/a; \tag{6.35}$$

$$t(t_2,X) = t_2 + (L-X)/a. \tag{6.36}$$

With these definitions the integrals of Equation 6.26 can be written as

$$\int_0^{X_{12}} dX \int_{t_o}^{t(t_1,X)} b(t)dt = b\int_0^{X_{12}} [t_1 + X/a - t_o]dX$$

$$= b\ [(t_1 - t_o)X_{12} + X_{12}^2/2a] \tag{6.37}$$

and

$$\int_0^{L-X_{12}} dX \int_{t_o}^{t(t_2,X)} b(t)dt = b\int_0^{L-X_{12}} [t_1+X'/a-t_o]dX'$$

$$= b\ [(t_2-t_o)(L-X_{12}) + (L-X_{12})^2/2a] \tag{6.38}$$

After some simplification and the use of Equation 6.34, the sum of these integrals can be written as

$$(ba/4)\ [2T(X+y) - (y-X)^2 + T^2] \tag{6.39}$$

where

$$X \equiv t_1-t_o; \tag{6.40}$$

$$y \equiv t_2-t_o; \tag{6.41}$$

$$T \equiv L/a. \tag{6.42}$$

Equation 6.29 can now be written in the form,

$$P(L) = b^2\int\int dX\ dy\ EXP\left\{-(ba/4)[2T(X+y)-(y-X)^2+T]\right\} \tag{6.43}$$

A number of conditions must be met for the limits of integration indicated in Equation 6.43. Since the break-up occurs between t_o and t_m

$$t_o < t_1 < t_m \quad , \quad 0 < X < t_m - t_o \equiv t \tag{6.44}$$

$$t_o < t_2 < t_m \quad , \quad 0 < y < t_m - t_o \equiv t \tag{6.45}$$

Because all the rod of length L must be relieved before time t_m,

$$t_u < t_m . \tag{6.46}$$

From Equation 6.33 it can be shown that

$$t_u = [L/a + t_1 + t_2]/2, \tag{6.47}$$

so that, from Equations 6.46 and 6.47

$$L/a + t_1 + t_2 < 2t_m , \; T + X + y < 2T. \tag{6.48}$$

Finally, the breaks at times t_1 and t_2 must be such that a relief wave from one does not arrive in time to prevent the other, i.e.

$$|t_1 - t_1| < L/a \quad , \quad |X - y| < T. \tag{6.49}$$

These conditions have been plotted in Figures 5 and 6 for t_1 and t_2. Since there is symmetry with respect to which break occurs first, the arbitrary condition

$$t_2 < t_1 \tag{6.50}$$

was added to reduce the range of integration.

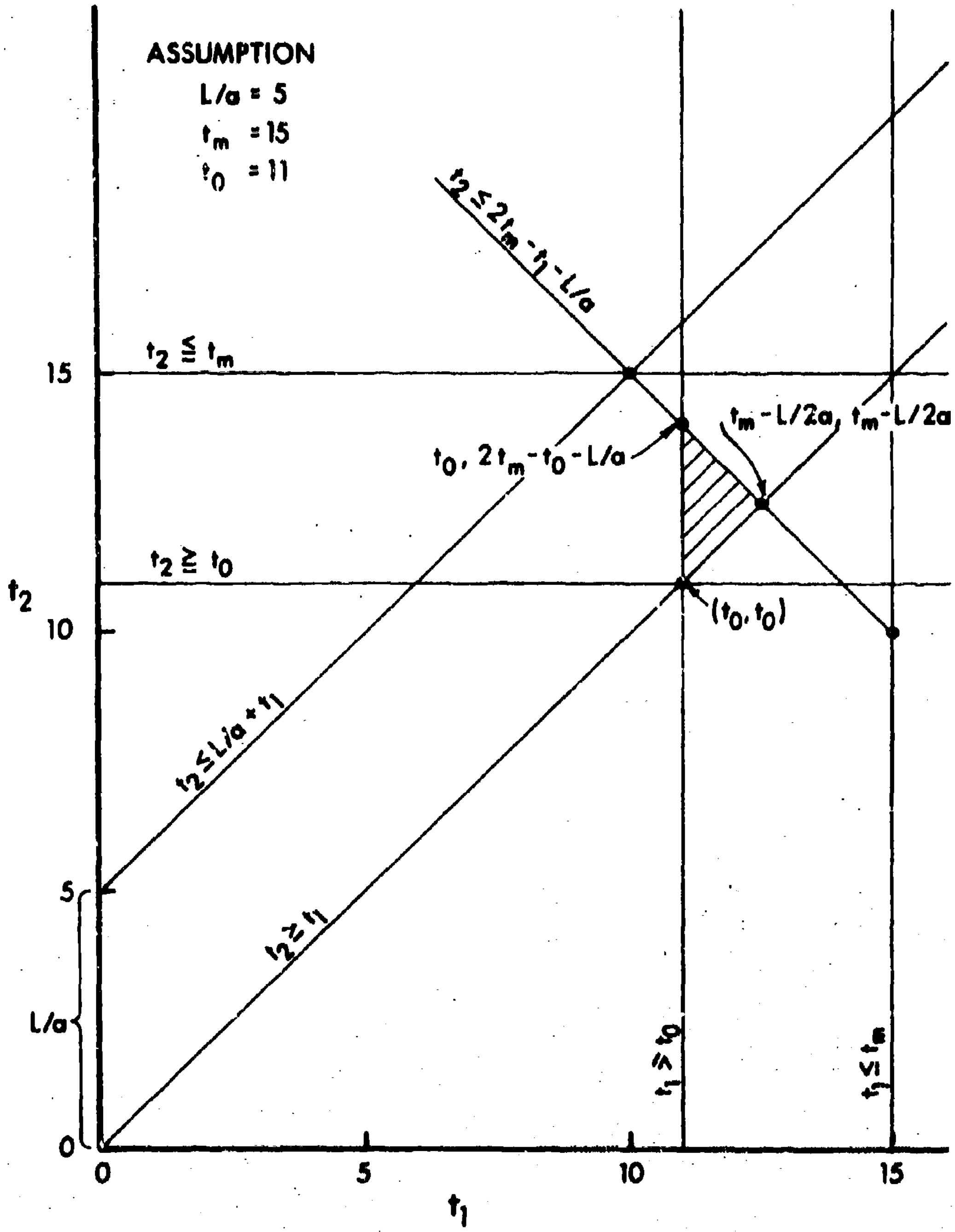

Figure 5. Graphical representation of the conditions of Eqs 6.44-6.50 for $L > a(t_m - t_o)$. The cross hatched area meets all the conditions.

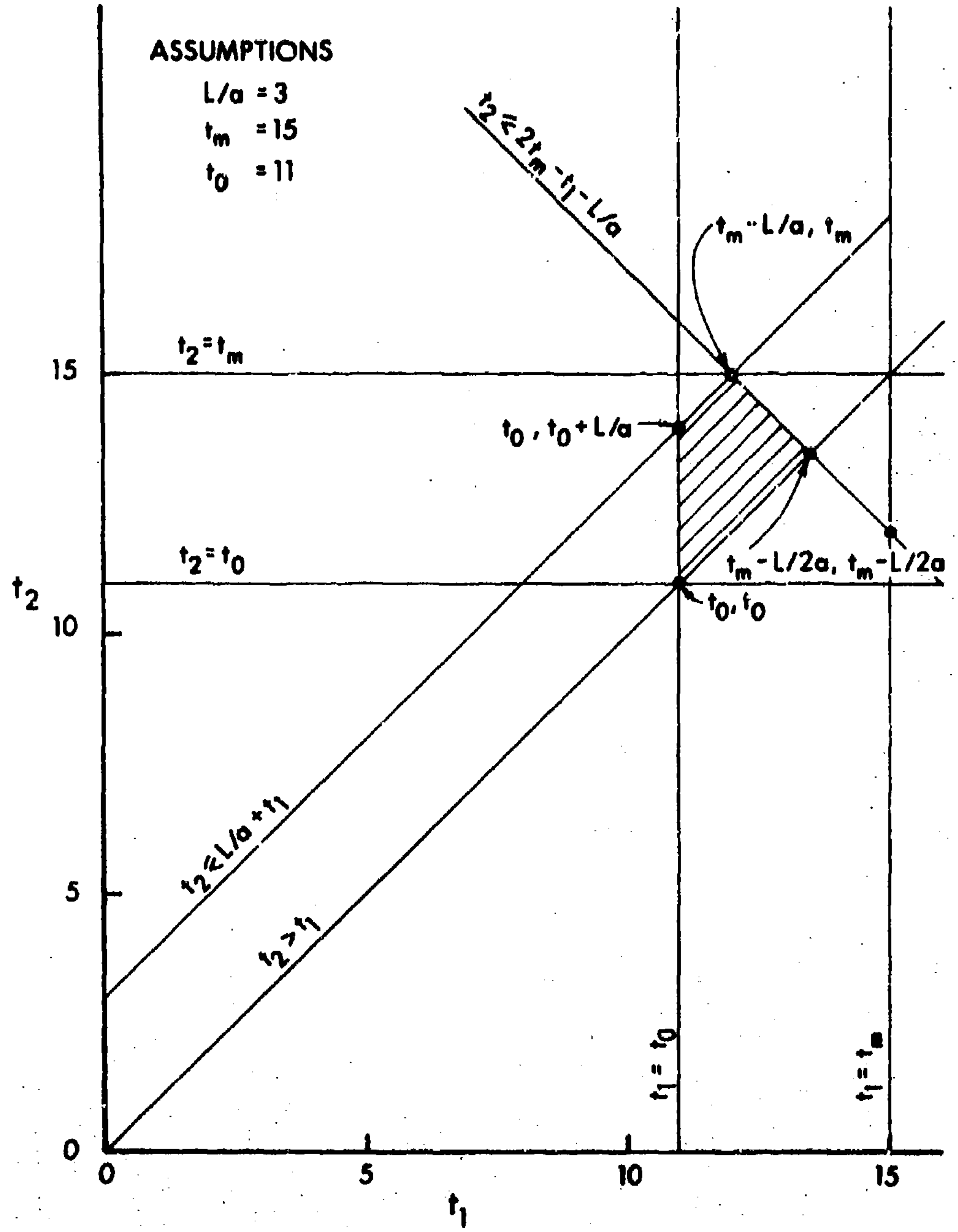

Figure 6. Graphical representation of the conditions of Equations 6.44-6.50 for $L<a(t_m-t_o)$. The cross hatched area meets all the conditions.

Unfortunately, the integral of Equation 6.43 cannot be expressed in simple terms. However, an upper limit can be assigned by noting that the exponential is always less than unity so that

$$P(L) < b^2 A(L), \tag{6.51}$$

where A(L) is just the area over which the integration is to be performed. By inspection of Figure 5 it is seen that the area of integration for $L > a(t_m - t_o)$ is one fourth of the square with edge of length $[2(t_m - t_o) - L/a]$. Therefore

$$A(L) = (1/4)[2(t_m - t_o) - L/a]^2. \tag{6.52}$$

In Figure 6, the area of integration is seen to be a trapezoid with altitude $\sqrt{2}L/2a$ and bases $\sqrt{2}(t_m - L/2a - t_o)$ and $\sqrt{2}(t_m - L/a - t_o)$. Therefore

$$A(L) = (1/2)(B_1 + B_2)h = (t_m - t_o)(L/a - 3L^2/4a^2) \tag{6.53}$$

Combining Equations 6.52 and 6.53 gives the result,

$$A(L) = (t_m - t_o)(L/a) - 3L^2/4a^2 \ , \ 0 < L < a(t_m - t_o) \tag{6.54}$$

$$= (1/4)[2(t_m - t_o) - L/a]^2 \ , \ a(t_m - t_o) < L < 2a(t_m - t_o).$$

It is seen that A(L) has a maximum at

$$L = 2a(t_m - t_o)/3 \tag{6.55}$$

and that A(L) meets the requirements that

$$A(0) = A[2a(t_m - t_o)] = 0, \tag{6.56}$$

as well as the requirement for continuity at $A[a(t_m - t_o)]$.

The maximum for P(L) should lie to the left of the maximum for $b^2A(L)$ because of the increasing influence of the exponential for larger values of L.

An experimental[5] plot of P(L) is shown in Figure 7. A graph of A versus L is shown in Figure 8. From Equation 6.56, if the maximum permissible length is determined from Figure 7 to be 3.8 cm and if the relief velocity is assumed to be 3750 m/sec,

$$2a(t_m-t_o) = L_{max} \tag{6.57}$$

$$t_m-t_o = 3.8\cdot10^{-2}/(2X3750) = 5X10^{-6}\ \text{sec.} \tag{6.58}$$

The ratio of maximum length to most probable length is roughly 3 from the experimental data and is also 3 from Equations 6.54 and 6.55.

The shape of the theoretical curve is not very good for small values of L, possibly because of the variation of a with distance from the break. The relief speed a is probably near 3750 m/sec at the break but less as the energy dissipation required of the relief process is increased, according to Equation 4.14.

The limits of integration for Equation 6.29 can be determined from Figures 5 and 6 by inspection. From Figure 5, for $L>a(t_m-t_o)$,

$$t_1<t_2<2t_m-t_1-L/a; \tag{6.59}$$

$$t_o<t_1<t_m-L/2a. \tag{6.60}$$

From Figure 6, for $0<L<a(t_m-t_o)$ and $t_o<t_1<t_m-L/a$,

$$t_1<t_2<L/a + t_1; \tag{6.61}$$

$$t_o<t_1<t_m-L/a. \tag{6.62}$$

From Figure 6, for $0<L<a(t_m-t_o)$ and $t_m-L/a<t_1<t_m-L/2a$,

$$t_1<t_2<2t_m-t_1-L/a; \tag{6.63}$$

$$t_m-L/a<t_1<t_m-L/2a. \tag{6.64}$$

[5]Robert L. Jameson of the Ballistic Research Laboratory, private communication, Nov 1977

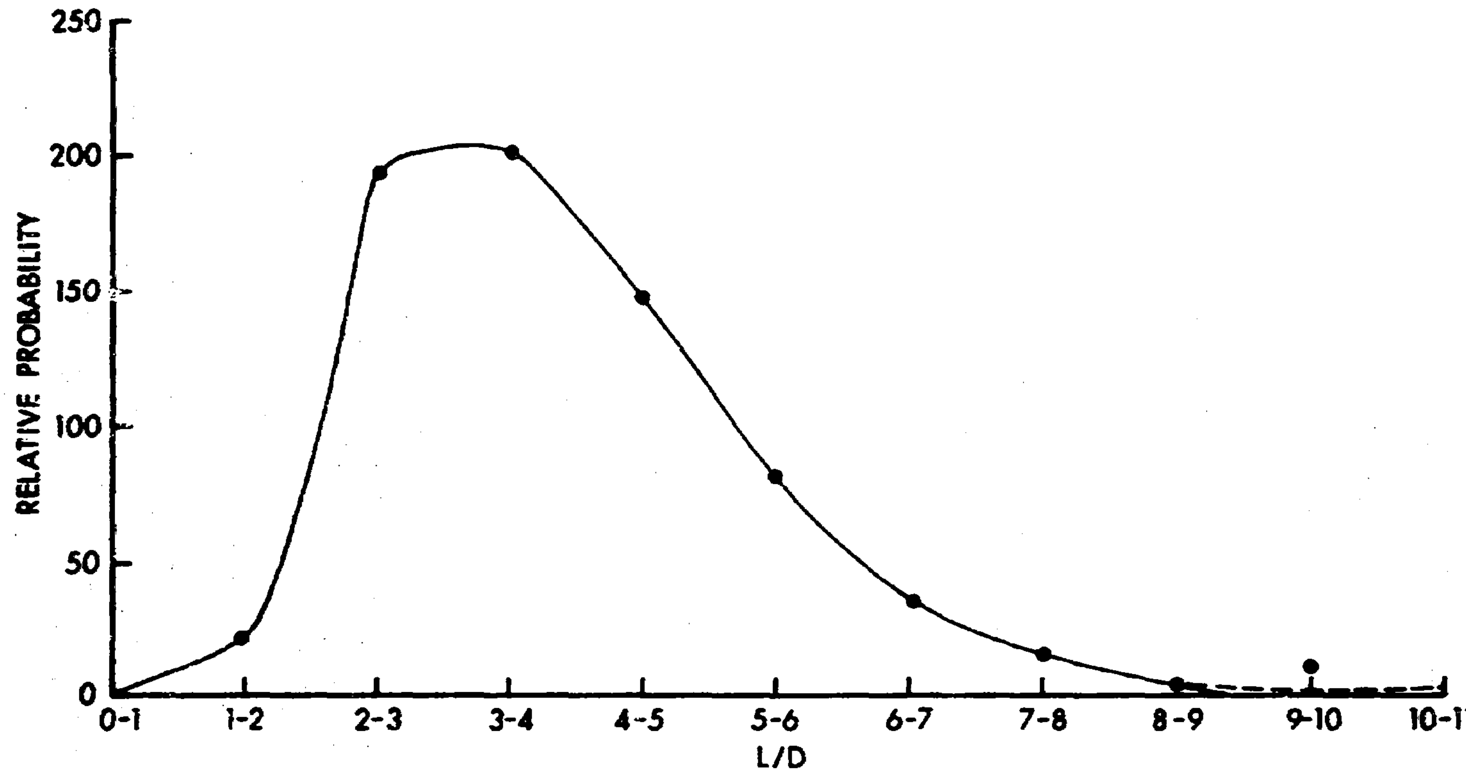

Figure 7. Experimentally observed distribution in length for a 70mm VIPER 43° 50' cone half angle. The cone is of copper and is 65mm in diameter and 0.97mm in thickness. The average particle diameter is 2.33mm.

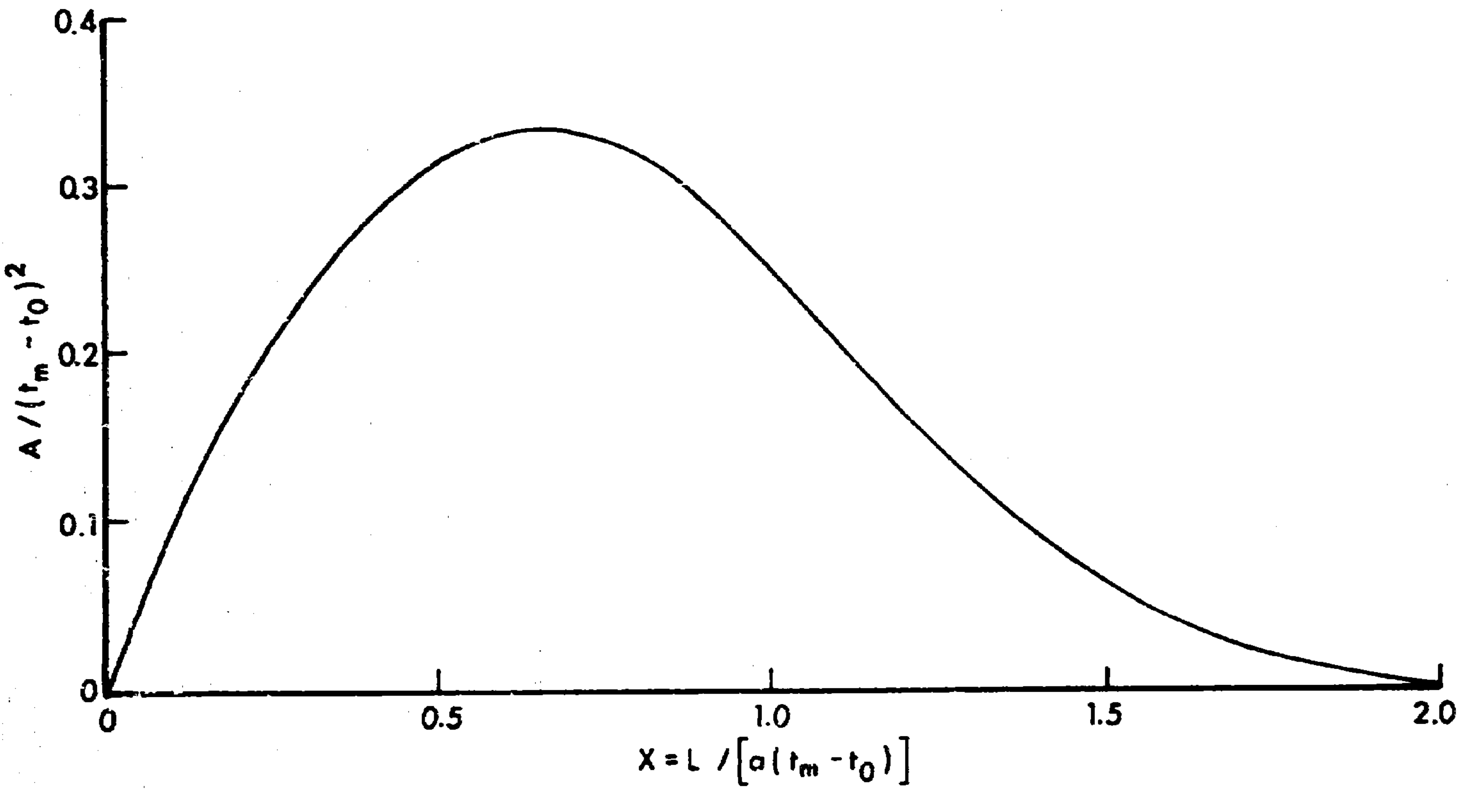

Figure 8. Theoretical distribution in length computed from Equation 6.54

These limits can be changed to suit Equation 6.43 by the substitutions of Equations 6.40-6.42.

For
$$T \geq t \equiv t_m - t_o, \tag{6.65}$$

$$X \leq y \leq 2t - X - T, \tag{6.66}$$

$$0 \leq X \leq t - T/2. \tag{6.67}$$

For $T \leq t$

$$X \leq y \leq T + X, \tag{6.68}$$

$$0 \leq X \leq t - T; \tag{6.69}$$

$$X \leq y \leq 2t - X - T; \tag{6.70}$$

$$t - T \leq X \leq t - T/2. \tag{6.71}$$

With these limits, Equation 6.43 may be written more explicitly as

$$P(T) = \int_0^{t-T} dX \int_X^{T+X} F(X,y)\,dy + \int_{t-T}^{t-T/2} dX \int_X^{2t-X-T} F(X,y)\,dy \tag{6.72}$$

when $0 \leq T \leq t$ and as

$$P(T) = \int_0^{t-T/2} dX \int_X^{2t-X-T} F(X,y)\,dy \tag{6.73}$$

when $t \leq T \leq 2t$, where

$$F(X,y) = \mathrm{EXP}\left\{-ba\left[T(X+y)/2 + T^2/4 - (X-y)^2/4\right]\right\}. \tag{6.74}$$

7. FURTHER CONSIDERATION OF JET BREAK-UP

It may seem reasonable from a physical viewpoint to remove the time restrictions on the onset and completion of jet break-up. Experience with pulling copper seems to justify the assignment of a narrow range between the strain necessary to start breakage in a collection of samples and the strain sufficient to insure breakage in every sample. Experimentally, there are no samples which do not break, and it is not reasonable to expect that a segment of a jet will be arbitrarily long after jet break-up and relief of the individual segments.

The process by which a segment, with a velocity gradient and two free ends, relieves itself is not well understood. If it is assumed, as shown in Figure 9, that the fragment of length L is relieved from the left end to the point X, which has a speed RX, the kinetic energy E_R for the relieved part can be written as

$$E_R = (M/2L)(L/2-X)R^2X^2 \tag{7.1}$$

The kinetic energy E_u of the unrelieved part is expressed as

$$E_u = (M/2L)\int_0^X R^2X^2dX = (M/6L)R^2X^3 \tag{7.2}$$

The total energy E of the left half of the segment is therefore given by the equation,

$$E = E_R+E_u = MR^2X^2/4-MR^2X^3/3L. \tag{7.3}$$

It will be assumed that this energy is removed by stretching of the unrelieved portion of the segment. The rate P at which work is done in stretching the unrelieved portion is

$$P = TRX \tag{7.4}$$

where T is the tension at the interface between the relieved and unrelieved regions and RX is the velocity of the material at the interface. In an instant of time dt, the work done dE can be written as

$$dE = Pdt = TRXdt. \tag{7.5}$$

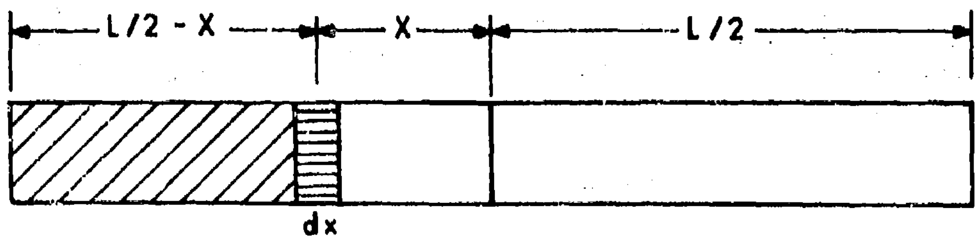

Figure 9. Hypothesized relief of a segment with an initial speed distribution U(X) = RX. The left portion of the segment, of length L/2-X, has been relieved by a wave located at X.

The energy loss by the segment is given by

$$(\partial E/\partial X)dX = (MR^2X/2 - MR^2X^2/L)dX = dE \qquad (7.6)$$

where E is given by Equation 7.3 and dX is the amount the interface moves in time dt. Equating tho results of Equations 7.5 and 7.6 gives the equation,

$$\frac{dX}{dt} = (TRX)/(MR^2X/2 - MR^2X^2/L) \qquad (7.7)$$

$$= SAL/(MRL/2 - MRX) = (S/\rho R)(L/2 - X) \qquad (7.8)$$

where the tension T has been written in terms of the stress S and the cross section C of the segment as SC and the density ρ has been defined as M/LC.

If the defination,

$$X' = (L/2) - X \qquad (7.9)$$

is made, Equation 7.8 becomes

$$dX/dt = S/\rho RX' \equiv 1/(AX') \qquad (7.10)$$

This is the speed which the interface (relief wave) must move to consume the available energy if the segment is relieved as hypothesized in Figure 9, where X′ is the distance from the left of the segment.

Following the pattern of Section 6, it is seen that the time for the relief wave to arrive at a point X from the end is given by

$$t(t_1.X) = t_1 + \int_0^X dX/U(X) = t_1 + \int_0^X AXdX = t_1 + AX^2/2 \qquad (7.11)$$

Likewise,

$$t(t_2,X') = t_2 + AX'^2/2. \qquad (7.12)$$

The waves meet at a time t_u and place X_{12} such that

$$t_1 + AX_{12}^2/2 = t_2 + A(L-X_{12})^2/2 = t_\mu \qquad (7.13)$$

so that

$$X_{12} = (t_2-t_1)/(AL) + L/2$$

and

$$t_u = (t_1+t_2)/2 + (t_2-t_1)^2/(2AL^2) + AL^2/8 . \qquad (7.14)$$

Proceeding as before the integrals of Equation 6.26 can be written as

$$I = b\int_0^{X_{12}} dX \int_{t_o}^{t(t_1,X)} dt + b\int_0^{L-X_{12}} dX' \int_{t_o}^{t(t_2,X)} dt, \qquad (7.15)$$

or more explicitly as

$$I = b\left\{(t_1-t_o)X_{12} + (A/6)X_{12}^2 + (t_2-t_o)(L-X_{12}) + (A/6)(L-X_{12})^2\right\}. \qquad (7.16)$$

This reduces to the expression,

$$I = b[(X+y)L/2 + AL^3/24 - (y-X)^2/2AL] \qquad (7.17)$$

where as before

$$X = t_1-t_o \qquad (7.18)$$

$$y = t_2-t_o \qquad (7.19)$$

$$t = t_m-t_c . \qquad (7.20)$$

The integration of EXP(-I) over the variables t_1 and t_2 is subject to several restrictions of the same kind as before, namely,

$$t_o \leq t_1 \leq t_m \quad , \quad 0 \leq X \leq t, \qquad (7.21)$$

$$t_o \leq t_2 \leq t_m \quad , \quad 0 \leq y \leq t, \qquad (7.22)$$

$$|t_2-t_1| \leq AL^2/2, \quad |y-X| \leq AL^2/2 \equiv T/2, \qquad (7.23)$$

$$t_m > t_u = (t_1+t_2)/2 + (t_2-t_1)^2/(2AL^2) + AL^2/8, \quad (7.24$$

$$2t > (X+y) + (X-y)^2/T + T/4. \quad (7.25)$$

Using the definition

$$T = AL^2 \quad (7.26)$$

Equation 7.17 can be written as

$$I = (bLT/2)\,[(X+y)/T + 1/12-(y-X)^2/T^2] \quad (7.27)$$

Equation 6.39 can be written for the constant relief velocity as

$$I = (baT^2/4)\,[2(X+y)/T + 1 - (y-X^2/T^2] \quad (7.28)$$

These functions (Equations 7.27 and 7.28) are somewhat similar in appearance but in one, Equation 7.27, T is defined as AL^2 whereas in Equation 7.28 it is defined as L/a. This means that longer segments are discriminated against when the relief wave speed varies as 1/AX. Small segments are also discrimminated against because the relief is very rapid near a break so that the next break is more likely to be distant. Thus the 1/AX assumption for the relief wave speed should result in a more narrow distribution than the assumption of a constant relief wave speed.

To illustrate the above points more specifically, from conditions described by Equations 6.47 and 6.48 and examination of Figure 6, it is noted that, as L approaches zero, the trapezoid, which represents the area over which the t_1 and t_2 integrations take place, approaches a height of $L/(\sqrt{2}a)$ and a length $\sqrt{2}(t-L/a)$. Therefore as L approaches zero

$$\text{Area} = Lt/a \quad (7.29)$$

This behavior at small L is exhibited by the curve of Figure 8 near the origin.

On the other hand, for the 1/AX hypothesis for the relief wave velocity, the condition of Equation 7.23 indicates that the area of integration, corresponding to that of Figure 6, has a width of $AL^2/2$. The length of the area, from Equation 7.24, in the limit as t_1 and t_2 are equal and approach t_m, is $\sqrt{2}\ (t-AL^2/8)$. In the limit as L goes to zero,

$$\text{Area} = AL^2/\sqrt{2}. \tag{7.30}$$

This behavior is characteristic of the experimental curve of Figure 7 near the origin. The hypothesis of a 1/AU relief velocity appears to yield a theoretical curve which is reasonable in shape near the origin and at large values of L, but the exact form of the curve is not known without evaluating the integral of EXP(-I), given by Equation 7.16 for various combinations of the parameters b, A, and t.

8. CONCLUSIONS

A simple physical model has been developed to explain jet break-up. It gives a fairly good description of the upper limit for particle length after break-up and predicts particle distribution with respect to length in a relative way. The model, unfortunately, makes use of material parameters which are not readily observable, such as the strength of a stretching jet, the limit to which the material can be worked by stretching without breaking, and the speed of a relief wave originating at the free end of a segment. It is probable that any model would have the same problem, because these parameters are fairly fundamental.

Some obvious suggestions for improving jets arise from the study. If the size of particles has an upper bound associated with the development of breaks at propitious times and places, why not design the jet to be weak at intervals whose distance is slightly less than the maximum permissible length? The jet could then consist of particles all of which are near maximum permissible length.

The time between onset and completion of jet break-up is estimated to be 5×10^{-6} seconds, and is inversely proportional to the speed of the relief wave propagating from a break. This corresponds to a strain of roughly 5% between onset and completion of break-up for the jet under study.

If the jet must break rather than stretch indefinitely, a more uniform jet will break into smaller pieces than a less uniform jet.

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